MATH 1650: FUNCTIONS AND THEIR REPRESENTATIONS

DEFINITION: Given two sets A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B. We think of the set A as being the set of **inputs** to the function while the set B is the set of **outputs** from the function.

EXAMPLES:

- Suppose we look at the process 'L' which matches each student in the class with the first letter of their first name. Is L a function? To determine if L is a function, we need to first identify the inputs, outputs, and make sure that each input is matched to only one output.

  The inputs to L are: students in the class

  The outputs from L are : letters of the alphabet

  Hence, to see if the process L is a function, we need to make sure that each: student in the class is matched with one and only one: letter of the alphabet.

  So, is L a function? Assuming each student has only one first name, then that name begins with only one letter, so L is a function.

- Suppose we reverse the process 'L' from the first example and create the process 'S' which matches each letter of the alphabet with the student in the class whose first name begins with that letter. Is S a function? As usual, let's first determine the inputs and outputs.

  The inputs to S are: letters of the alphabet

  The outputs from S are: students in the class

  Hence, to see if the process S is a function, we need to make sure that each: letter of the alphabet is matched with one and only one: student in the class.

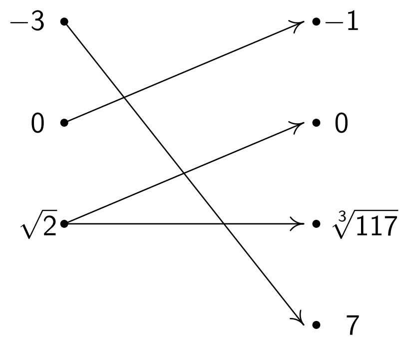
  So, is S a function? Well, that depends on the class. There could be some letters that get matched with no students at all! There could be two or more students whose first names start with the same letter! So unless we have 26 students in the class, and each student's first name begins with a different letter, S is not a function.

- Consider the process T that maps the time on a particular day to the temperature as registered by the Burke Lakefront Airport. Is T is a function? What about the reverse process?

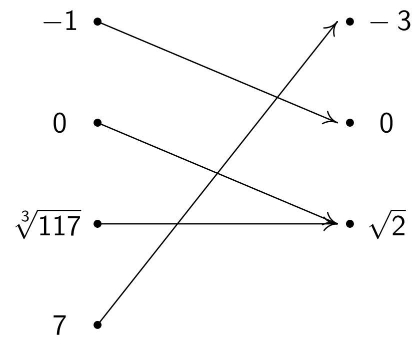
  The inputs to T are the times during the day and the outputs from T are the temperatures registered by the Burke Lakefront Airport. Since there is only one given temperature for each time, T is a function. The reverse process would match temperatures recorded at the Airport back to the time at which it was recorded. If the same temperature is recorded at different times, the reverse process would not be a function.

Another way to describe functions is by visualizing them using 'Mapping Diagrams' as shown below. Here, the **inputs** to the mapping are shown on the **left** and they are matched to their corresponding **outputs** on the **right**.

- The mapping f:



- The mapping g:



- On the mapping f, the input -3 is matched to 7.
- On the mapping g, the input 0 is matched to $\sqrt{2}$.

Remember, to be a **function**, a mapping must match each input to **only one** output.

- Is the mapping f a function? Why or why not?

  No, f is not a function since the input $\sqrt{2}$ is matched with more than one output: 0 and $\sqrt[3]{117}$.

- Is the mapping g a function? Why or why not?

  Yes, g is a function since each input is matched with only one output. Note that it is totally fine that two different inputs here, 0 and $\sqrt[3]{117}$ have the same output.

DEFINITION:

- The **domain** of a function is the set of all inputs to the function.
- The **range** of a function is the set of all outputs from the function.
- If the range is a subset of the real numbers:
  - The **minimum** of a function is the smallest number in the range (if it exists.)
  - The **maximum** of a function is the largest number in the range (if it exists.)

EXAMPLE:

- State the domain and range of the function g.

  domain of g is $\{-1, 0, \sqrt[3]{117}, 7\}$. The range of g is $\{-3, 0, \sqrt{2}\}$.

- Find the maximum and minimum of g.

  The maximum of g is $\sqrt{2}$; the minimum of g is -3.

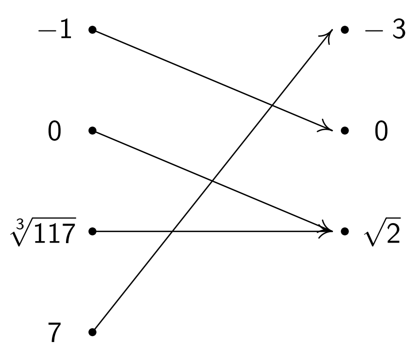
FUNCTION NOTATION

DEFINITION: Suppose the process f is a function and x is in the domain of f. The notation '$f(x)$' which is read 'f **of** x' represents the **output** from the process f using the **input** x. That is:

$$f(\text{input}) = \text{output}$$

EXAMPLE:

- Let L be the process by which each student is matched with the first letter of their first name. Suppose a student named Emmy Noether is in the class. What is $L(\text{Emmy Noether})$? $L(\text{Emmy Noether}) = \text{E}$.
- Consider the function g below:



Find and simplify the following:

1. $g(0) = \sqrt{2}$
2. $g(7) = -3$
3. $g(0) + g(7) = \sqrt{2} + (-3) = \sqrt{2} - 3$
4. $g(0+7) = g(7) = -3$
5. To solve $g(x) = 0$, we are looking for the **inputs** to g which give an **output** of 0. Hence, $g(x) = 0$ when $x = -1$.
6. To solve $g(t) = \sqrt{2}$, we are looking for the **inputs** to g which give an **output** of $\sqrt{2}$. Hence, $g(t) = \sqrt{2}$ when $t = 0$ or $t = \sqrt[3]{117}$.

- Suppose T is the function which maps the time on January 1, 2019 to the temperature at Burke Lakefront Airport. More specifically, let $T(t)$ denote the temperature (in degrees Fahrenheit) t hours after 6 AM.

1. $T(0) = 45$ means that at 6 AM (0 hours after 6 AM), the temperature is 45°F.
2. At noon, the temperature was recorded as 36°F $\iff$ $T(6) = 36$.
3. The solutions to $T(t) = 40$ are the times on January 1, 2019 when the temperature was 40°F.
4. The minimum of T is the lowest temperature of the day while the maximum of T would be the highest temperature of the day.

ALGEBRAIC REPRESENTATIONS OF FUNCTIONS

EXAMPLE:

- Let F be the process which takes a real number and performs the following sequence of operations:
  - Step 1: square the number
  - Step 2: add 1 to the result of Step 1.

  1. $F(3) = (3)^2 + 1 = 10$
  2. $F(-2) = (-2)^2 + 1 = 5$
  3. $F(x) = x^2 + 1$
  4. $x^2 + 1 = 4$ gives $x^2 = 3$ or $x = \pm\sqrt{3}$

- Let $g(t) = 2t^2 - 3t + 1$. Find and simplify:

  1. $g(0) = 2(0)^2 - 3(0) + 1 = 1$
  2. $g(-1) = 2(-1)^2 - 3(-1) + 1 = 6$
  3. $g(3a) = 2(3a)^2 - 3(3a) + 1 = 18a^2 - 9a + 1$
  4. $3g(a) = 3(2(a)^2 - 3(a) + 1) = 6a^2 - 9a + 3$
  5. $$\begin{aligned} g(x+2) &= 2(x+2)^2 - 3(x+2) + 1 \\ &= 2(x^2 + 4x + 4) - 3x - 6 + 1 \\ &= 2x^2 + 8x + 8 - 3x - 6 + 1 \\ &= 2x^2 + 5x + 3 \end{aligned}$$
  6. $$\begin{aligned} g(x) + 2 &= (2(x)^2 - 3(x) + 1) + 2 \\ &= 2x^2 - 3x + 3 \end{aligned}$$
  7. $$\begin{aligned} g(x) + g(2) &= (2(x)^2 - 3(x) + 1) + (2(2)^2 - 3(2) + 1) \\ &= 2x^2 - 3x + 1 + 3 \\ &= 2x^2 - 3x + 4 \end{aligned}$$

DEFINITION: Given an equation involving two variables (say x and y) we say the equation describes 'y as a function of x' if each choice of x produces (at most) one resulting value of y.

EXAMPLE:

- Determine if the equation $x^3 - y^2 = 1$ describes y as a function of x.

  To determine if 'y' is a function of 'x,' we need to determine if x produces only one y.

  To help us answer this question, we solve for y in terms of x:

$$\begin{aligned} x^3 - y^2 &= 1 \\ -y^2 &= 1 - x^3 \\ y^2 &= x^3 - 1 \\ y &= \pm\sqrt{x^3 - 1} \end{aligned}$$

  Here, y is **not** a function of x. For example, the value $x = 2$ produces **two** $y-$values: $y = \pm\sqrt{7}$.

- Determine if the equation $xy^3 = 8$ describes y as a function of x. We solve for y in term of x:

$$\begin{aligned} xy^3 &= 8 \\ y^3 &= \frac{8}{x} \\ y &= \sqrt[3]{\frac{8}{x}} \\ y &= \frac{2}{\sqrt[3]{x}} \end{aligned}$$

  As long as $x \neq 0$, each x produces only one y. If $x = 0$, then the original equation $xy^3 = 8$ reduces to $(0)y^3 = 8$ or $0 = 8$ which is never true, so we can ignore this case. Hence, y is a function of x.

DEFINITION: If an equation describes y as a function of x, then there is a function f so that $y = f(x)$. In this case we say y is the **dependent** variable and x is the **independent** variable.

QUESTION: Which of the equations above describe x as a function of y?

- Solving $x^3 - y^2 = 1$ for x produces $x = \sqrt[3]{y^2 + 1}$. Since each y produces only one x, x is a function of y.
- Soving $xy^3 = 8$ for x produces $x = \frac{8}{y^3}$. For all $y \neq 0$, each y produces one x, so x is a function of y. (Note that substituting $y = 0$ into $xy^3 = 8$ produces $0 = 8$ again so we ignore this value.)

GEOMETRIC REPRESENTATIONS OF FUNCTIONS

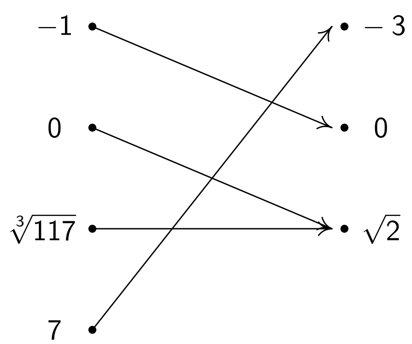
Consider the function g described by the mapping diagram below on the left.

We can more succinctly describe g as a set of *ordered pairs* of the form (input, output):

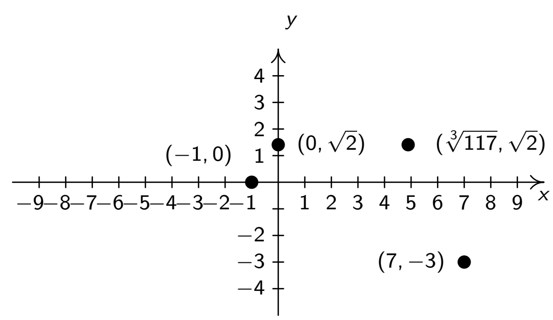
$$g = \{(-1, 0), (0, \sqrt{2}), (\sqrt[3]{117}, \sqrt{2}), (7, -3)\}$$

Plotting these points results in the **graph** of g. Graph g below on the right on the xy-plane.

A mapping diagram of g



The graph of g

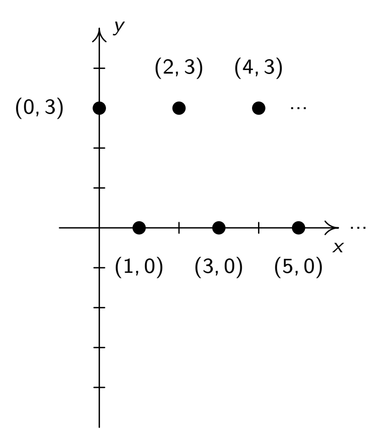


In general, given a function f, we may describe f as a set of ordered pairs:

$$\{(x, f(x)) \,|\, \text{for } x \text{ in the domain of } f\} = \{(\text{input}, \text{output})\}$$

EXAMPLE: Consider the set of points $F = \{(0, 3), (1, 0), (2, 3), (3, 0), (4, 3), (5, 0), (6, 3), ...\}$ in the xy-plane.

1. F represents y as a function of x since each x is matched with only one y. One way to see this is that no two different points have the same x-coordinate.
2. We see $(0, 3)$ belongs to F, so $F(0) = 3$.
3. To solve $F(x) = 0$, we look for the $x-$values for which the $y = 0$. We get $\{1, 3, 5, ...\}$
4. The domain of F is $\{0, 1, 2, 3, 4, 5, ...\}$ and the range of F is $\{0, 3\}$.
5. The maximum of F is 3 and the minimum of F is 0.
6. The graph of F is below: the '...' mean the graph continues off to the right in a similar fashion.



The graph of $y = F(x)$.

EXAMPLE: Which of the following sets of points in the xy-plane represent y as a function of x?

HINT: Try to find an equation that relates the x and y coordinates.

1. $h = \{(a, b) \mid b = |a|\}$

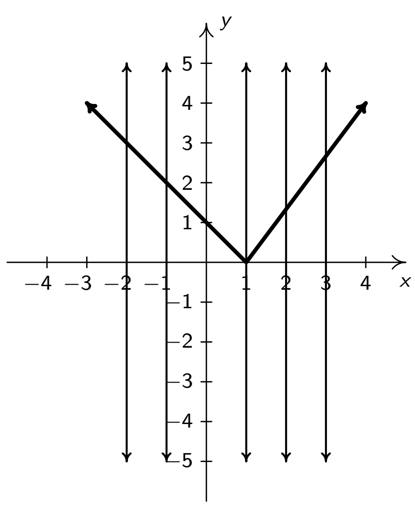
   Here $x = a$ and $y = b$, so we have this set or ordered pairs described by the equation $y = |x|$. Each x produces only one y so y is a function of x.

2. $G = \{(2t^2, t) \mid t \text{ is a real number}\}$

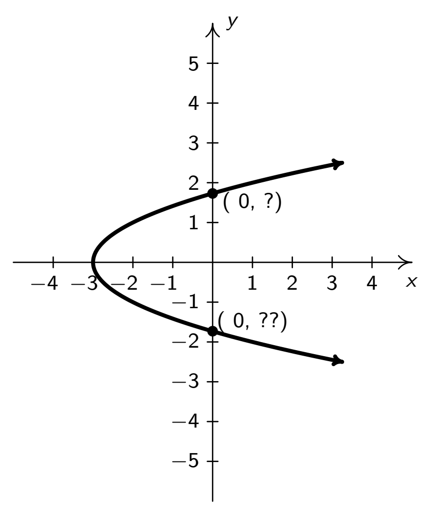
   Here $x = 2t^2$ and $y = t$ so $x = 2y^2$. Solving for y, we get $y = \pm\sqrt{\dfrac{x}{2}}$, so, for instance, when $x = 2$, $y = \pm 1$ so y is not a function of x.

EXAMPLE: Consider the sets of points S and T plotted below on the xy-plane.

- Graph of the set S.



- Graph of the set T.



1. Does the set S represent y as a function of x?

   Yes! for each x, there is only one y coordinate (no two points on the graph have the same x-coordinate.)

   Geometrically, each vertical line crosses the graph at most once, so the graph passes the VLT.

2. Does the set T represent y as a function of x?

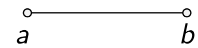
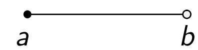
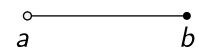
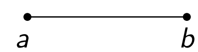
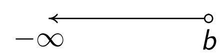
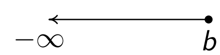
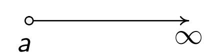
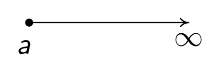
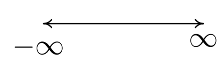
   No! For example, there are two points on the graph with an x-coordinate of 0 (two y-intercepts.)

   Geometrically, the vertical line $x = 0$ (a.k.a. the $y-$axis) crosses the graph more than once.

QUESTION: How would we determine if a graph on the xy-plane represents x as a function of y?
We would use **horizontal** lines instead of vertical lines.

REVIEW OF INTERVAL NOTATION

Let a and b be real numbers with $a < b$.

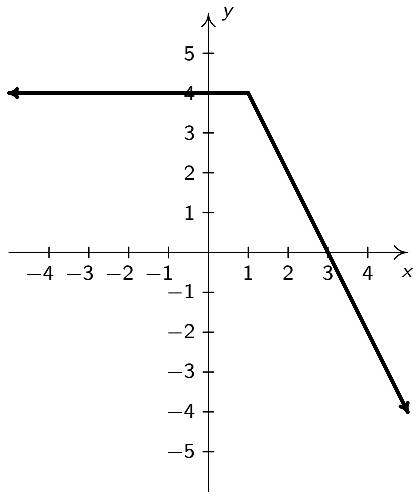
Set of Real Numbers	Interval Notation	Region on the Real Number Line
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$	$(a, b]$	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \leq b\}$	$(-\infty, b]$	
$\{x \mid x > a\}$	(a, ∞)	
$\{x \mid x \geq a\}$	$[a, \infty)$	
$\mathbb{R}$	$(-\infty, \infty)$	

Notational Rules:

- It's always the smaller number (or $-\infty$) first.
- If a number is not included, we use a parenthesis: '(' or ')' ('∞' and '$-\infty$' are never included!).
- If the number is included, we use a bracket: '[' or ']'.

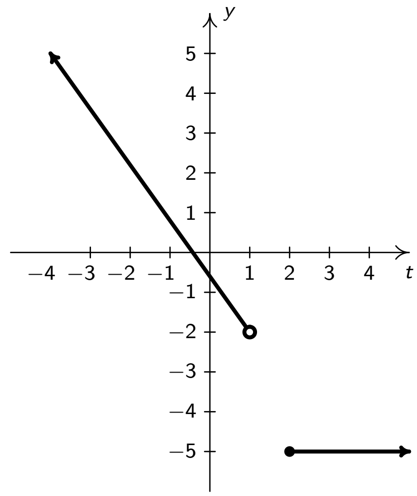
NOTE: When applicable, we will be using interval notation, when describing sets of real numbers.

EXAMPLE: Use the graphs of the functions below to answer the indicated questions.



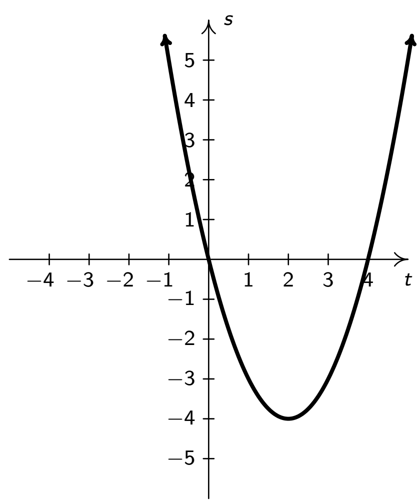
$y = C(x)$

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, 4]$
- Minimum: none
- Maximum: 4
- Find $C(-1.1) = 4$.



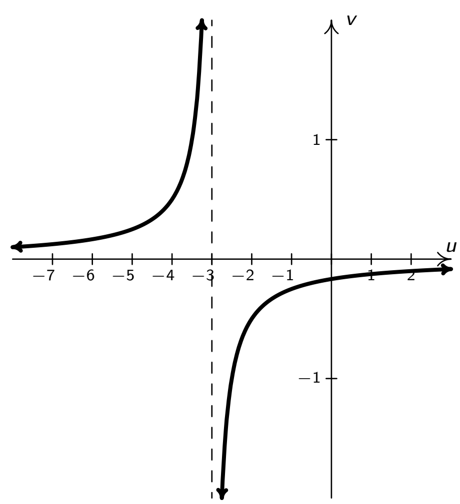
$y = N(t)$

- Domain: $(-\infty, 1) \cup [2, \infty)$
- Range: $\{-5\} \cup (-2, \infty)$
- Minimum: -5
- Maximum: None
- $N(2) = -5$.



$s = h(t)$

- Domain: $(-\infty, \infty)$
- Range: $[-4, \infty)$
- Minimum: -4
- Maximum: None
- Solve $h(t) = 0$ when $t = 0$ or $t = 4$.

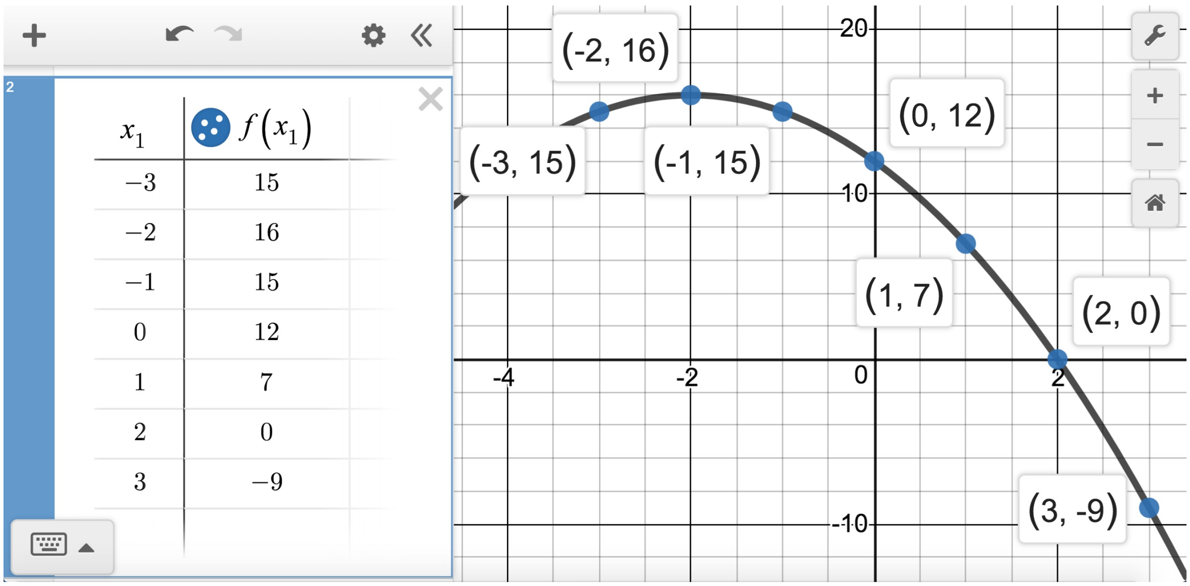


$v = r(u)$

- Domain: $(-\infty, -3) \cup (-3, \infty)$
- Range: $(-\infty, 0) \cup (0, \infty)$
- Minimum: None
- Maximum: None
- $r(u) > 0$ when $u < -3$.
  Using interval notation, $(-\infty, -3)$

EXAMPLE: Graphing $f(x) = 12 - 4x - x^2$ on the xy-plane.

x	$y = f(x)$	$(x, f(x))$
-3	15	$(-3, 15)$
-2	16	$(-2, 16)$
-1	15	$(-1, 15)$
0	12	$(0, 12)$
1	7	$(1, 7)$
2	0	$(2, 0)$
3	-9	$(3, -9)$



From the graph, determine the:

- domain: $(-\infty, \infty)$
- range: $(-\infty, 16]$
- maximum: 16
- minimum: None

EXAMPLE: Suppose $T(t)$ represents the temperature in degrees Fahrenheit of piping hot coffee t minutes after it is served.

1. T is a function since the coffee can only have one temperature at any given time.

2. $T(15)$ gives the temperature of the coffee 15 minutes after it is served.

3. The solutions to $T(t) = 120$ tell us when (how many minutes after the coffee is served) the coffee is 120°F.

4. The domain of T would likely be $t \geq 0$ until the coffee is consumed.

5. To know the range of T, we'd have to know how hot the coffee is when served and the room temperature.

6. I'd expect the temperature curve to look like a rapid decrease after the coffee is served and leveling off at room temperature.